246. Proposed by ALBERT A. BENNETT, Princeton University.

Prove that

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]$$

is an integer for every positive integral value of n, whenever a is an odd integer, positive or negative, and $b \equiv 1 \pmod{4}$.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Taking out the factor

$$\frac{a+\sqrt{b}}{2}-\frac{a-\sqrt{b}}{2}=\sqrt{b},$$

we have

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]=\sum_{s=0}^{n-1}\left(\frac{a+\sqrt{b}}{2}\right)^{n-s-1}\left(\frac{a-\sqrt{b}}{2}\right)^s.$$

Let

$$\frac{a+\sqrt{b}}{2}=x, \qquad \frac{a-\sqrt{b}}{2}=y.$$

Then this expression is

(I) $$(x^{n-1}+y^{n-1})+xy(x^{n-3}+y^{n-3})+x^2y^2(x^{n-5}+y^{n-5})+\cdots+L;$$

where $L=x^{(n-1)/2}y^{(n-1)/2}$ if n is odd; and $L\equiv x^{(n-2)/2}y^{(n-2)/2}(x+y)$ if n is even.

Let $a=2k+1$, $b=4l+1$, k and l being integers. Then

$$xy=\frac{a^2-b}{4}=k^2+k-l,$$

an integer, so that any positive integral power of xy is an integer.

Also $x+y=a$ and $x^2+y^2=(x+y)^2-2xy=a^2-2(k^2+k-l)$ are integers. Further,

$$x^p+y^p=(x+y)^p-\Sigma\frac{p!}{s!(p-s)!}x^sy^s(x^{p-2s}+y^{p-2s});$$

so that if x^p+y^p is integral for a positive integral value of p, it is integral for a value of p greater by 2; but it is integral for $p=1$ and for $p=2$, and so is integral for any positive integral value of p.

Thus each term of (I) is integral, and I is an integer.

Also solved by ELBERT H. CLARKE.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO INFINITESIMAL METHODS IN GEOMETRY.

BY E. B. WILSON, Massachusetts Institute of Technology.

The old method of using infinitesimals to find by geometric considerations various properties of figures was given a prominent place by W. E. Byerly in his *Differential Calculus*, but has been almost wholly discarded by recent writers. It is a question worthy of discussion whether the present neglect of the method is entirely wise. An example or two may be given.

1. *Consider the problem of the maximum triangle inscribed in a convex curve, or the minimum triangle circumscribed about the curve.*

If ABC is the maximum triangle inscribed in the curve, a displacement of any vertex, say C, along the curve, can only change the area of the triangle by an infinitesimal of higher order than the first, and hence the altitude from C to AB must change only by an infinitesimal of higher order. This means that the tangent to the curve at C must be parallel to the side AB. We have, therefore, the proposition that the maximum triangle is that for which the tangents to the curve at the vertices are parallel to the opposite sides of the triangle. Incidentally, the geometry of the figure shows that the triangle formed by the three tangents has its three sides bisected by the points of tangency.

If the minimum triangle circumscribed about the curve is desired, let PQR be the triangle. The displacement of the point of tangency of one side must change the area of the triangle by an infinitesimal of higher order. A consideration of the small triangles added to and subtracted from PQR by the change of one side to a neighboring point of tangency, shows at once that the triangle PQR has its sides bisected by their points of tangency. Incidentally the geometry of the construction shows that the inscribed triangle formed by joining the three points of tangency has its sides parallel to the sides of PQR.

It is thus seen that the solutions for the maximum inscribed triangle and the minimum cricumscribed triangle are identical in so far as the determination of three points on the curve is concerned. Is there any *easy* way of reaching these results by the exclusively analytical methods now in vogue?

As applied to the circle, it is a matter of the simplest plane geometry to show that the above construction calls for equilateral triangles. Is there any easier way than this for finding the maximum and minimum triangles for the special case of the circle?

Any ellipse may be regarded as the orthogonal projection of a circle. In such a projection areas are multiplied by a constant, the cosine of the angle between the planes. The eccentric angle in the ellipse is identical with the central angle in the circle. Hence we see that the maximum triangle inscribed in the ellipse is any triangle whose vertices are equally spaced along the ellipse when measured by the eccentric angle (or by the sectorial area; *i. e.*, the central radii to the vertices of the triangle divide the ellipse into three sectors of equal area). The minimum triangle about the ellipse needs no special attention, because of the general theorem mentioned above.

2. *Consider the problem of finding the point P in the plane of three points ABC such that the sum of the distances $PA + PB + PC$ shall be a minimum.*

If this problem is solved by the straightforward analytical method, the analysis appears complicated, owing to the three radicals used to express the distances.

The geometric solution is easy. Let P be the point. Displace P along a line perpendicular to one of the lines PA, PB, PC, say PC. Then PC changes by an infinitesimal of the second order, and $PA + PB$ must change by an

infinitesimal of higher order than the first; that is, $PA + PB$ is constant up to and including the first order. Hence (as shown by the laws of reflection for light, or from the properties of the ellipse), PA and PB must make equal angles with PC. By symmetry, the angles subtended at P by the three sides of the triangle ABC must all be 120°.

It is now easy to construct the point P. Draw upon two sides of the triangle arcs of 120°. Their intersection determines P, and the arc of 120° on the third side of the triangle passes through P. (The construction breaks down when one angle of the triangle is greater than 120°, and in this case the vertex of that angle is the solution of the problem.)

I should be glad to see other analytic solutions of the problem which are as simple. See the solution by Dunham Jackson in the January 1917 MONTHLY, which appeared after this note was sent to the editor.

II. Relating to a Problem in Minima Discussed by Professor Dunham Jackson in the January, 1917, Number (p. 46) of this Monthly.

By Roger A. Johnson, Western Reserve University, Cleveland, Ohio.

The problem is to determine a point the sum of whose distances from the vertices of a given triangle shall be a minimum.

Professor Jackson has apparently overlooked the fact that this problem admits an elementary geometric solution due to Steiner (*Collected Works*, Vol. II, p. 729). The following is a development of this solution; for more detailed treatment and historical notes, see Emmerich, *Die Brocard'sche Gebilde*, § 42.

Lemma I. The sum of the distances to the sides of an equilateral triangle, from a point inside the triangle, is constant and equal to the altitude. Moreover, if from any point the perpendiculars to the sides of the equilateral triangle make angles of 120° (and not 60°) with one another, the point is inside the triangle.

Lemma II. Let ABC be a triangle, each angle of which is less than 120°. There is a point P, and only one, such that $\angle BPC = \angle CPA = \angle APB = 120°$.

Lemma III. Through A, B, C, draw MN, NL, LM, perpendicular to PA, PB, PC, respectively. Then LMN is an equilateral triangle and P is inside it.

THEOREM. *If Q is any point of the plane, other than P, then*

$$QA + QB + QC > PA + PB + PC.$$

For $QA + QB + QC$ is greater than the sum of the perpendiculars from Q to MN, NL, LM; and the sum of these perpendiculars is greater than or equal to $PA + PB + PC$, according as Q lies outside or inside triangle LMN.

The point P is one of two points called *isogonic centers*, which have numerous interesting properties (cf. Emmerich, *l. c.*). If any angle of the triangle is greater than 120°, it falls outside the triangle and therefore fails to yield a minimum. As Professor Jackson indicates, it is not hard in this case to show that the vertex of the obtuse angle is the desired point. To do this by the same method we have